Nonlocal Graph Morphology

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Abstract

Mathematical morphology (MM) operators can be defined in terms of algebraic sets or as partial differential equations. In this paper, we introduce a novel formulation of MM over weighted graphs of arbitrary topology. The proposed framework recovers local algebraic and PDEs-based formulations of MM and introduces nonlocal configurations. This enables to PDEs-based methods to process any discrete data that can be described by a graph such as high dimensional data defined on irregular domains.

Index Terms— Nonlocal, graphs, morphology.

1 Introduction

Two formulations of morphological operators can be found in literature: the algebraic formulation and the Partial Differential Equations (PDEs) formulation. Morphological algebraic flat dilation δ and erosion ε of a function $f^0{:}\mathbb{R}^n{\to}\mathbb{R}$ are usually formulated by: $\delta_B(f^0)(x) = \sup\{f^0(x{+}y) : y{\in}B\}$ and $\varepsilon_B(f^0)(x) = \inf\{f^0(x{+}y) : y{\in}B\}$ with B a compact convex symmetric set (called structuring element). By using structuring sets $B{=}\{x : \|x\|_p{\leq}1\}$, the general PDEs generating these *flat* dilations and erosions [2] are as follows:

$$\frac{\partial\delta(f)}{\partial t}{=}\partial_t f{=}{+}\|\nabla f\|_p \text{ and } \frac{\partial\varepsilon(f)}{\partial t}{=}\partial_t f{=}{-}\|\nabla f\|_p \tag{1}$$

where f is a modified version of f^0, ∇ is the gradient operator, $\|.\|_p$ corresponds to the $\mathcal{L}_p$-norm, and one has the initial condition $\partial_{t=0}f{=}f^0$. With different values of p, one obtains different structuring elements: a rhombus for $p{=}\infty$, a disc with $p{=}2$ and a square with $p{=}1$ [2].

Whatever the chosen formulation (algebraic or PDEs), there is actually no satisfying formulation of MM for the processing of multivariate unorganized data. The algebraic formulation has drawbacks for multivariate data (no natural ordering for vectors to form a complete lattice [1]) and the PDEs formulation has drawbacks for unorganized data (spatial discretization is difficult). In addition, whatever the formulation, both only consider local interactions on the data while nonlocal schemes have recently received a lot of attention for arbitrary data processing [3, 4]. In the spirit of [4], we propose to consider a discrete version of continuous MM over weighted graphs that naturally enables nonlocal processing of any multivariate data living on any domain.

2 Operators on graphs

We consider the general situation where any discrete domain can be viewed as a weighted graph. Let $G{=}(V,E,w)$ be a *weighted graph* composed of a finite set of *vertices* V and a finite set of *weighted edges* $E{\subseteq}V{\times}V$. An edge $(u,v){\in}E$ connects two *adjacent* or *neighbor* vertices u and v of V. The set of neighbors of a vertex u is denoted by $N(u){=}\{v{\in}V\backslash\{u\} : (u,v){\in}E\}$. The graph can be associated with a weight function $w{:}E{\to}\mathbb{R}^+$. Graphs are assumed to be simple, connected and undirected, implying that the weight function w is symmetric. For the sake of simplicity, $w(u,v)$ will be denoted by w_{uv} and notation $v{\sim}u$ will mean that vertices u and v are adjacent. Let $f{:}V{\to}\mathbb{R}$ be a discrete real-valued function that assigns a real value $f(u)$ to each vertex $u{\in}V$. Moreover, each vertex $u{\in}V$ can be assigned with a feature vector $F(f^0,u){\in}\mathbb{R}^m$ that is used to compute weights:

$$g_1(u,v){=}\exp(-\|(f^0,u) - F(f^0,v)\|^2/\sigma^2) \text{ with } \sigma{>}0 \ ,$$
$$g_2(u,v){=}(\|F(f^0,u) - F(f^0,v)\| + \epsilon)^{-1} \text{ with } \epsilon{>}0, \epsilon{\to}0 \ .$$

A typical feature vector is $F(f^0,u){=}f^0(u)$ (local processing) or $F(f^0,u) = F_\tau(f^0,u) = \{f^0(v) : v{\in}N_\tau(u) \cup \{u\}\}$ with $N_\tau(u) = \{v{\in}V\backslash\{u\} : \|u-v\|{\leq}\tau\}$ (nonlocal processing). We denote $\mathcal{A}$ as the set of connected vertices with $\mathcal{A}{\subset}V$ such that for all $u{\in}\mathcal{A}$, there exists a vertex $v{\in}\mathcal{A}$ with $(u,v){\in}E$. We denote by $\partial^+\mathcal{A}$ and $\partial^-\mathcal{A}$, the *external* and *internal* boundary sets of $\mathcal{A}$, respectively. For a given vertex $u{\in}V$:

$$\partial^+\mathcal{A} = \{u{\in}\mathcal{A}^c : \exists v{\in}\mathcal{A} \text{ with } (u,v){\in}E\} \text{ and}$$
$$\partial^-\mathcal{A} = \{u{\in}\mathcal{A} : \exists v{\in}\mathcal{A}^c \text{ with } (u,v){\in}E\} \ , \tag{2}$$

where $\mathcal{A}^c{=}V \setminus \mathcal{A}$ is the complement of $\mathcal{A}$.

2.1 Weighted morphological differences

All the basic operators considered in this paper are defined from the difference operator or the discrete derivative. In this work, we consider a definition that allows the expressions of discrete weighted gradient on graphs. The *weighted difference operator* [4] $d_w{:}\mathcal{H}(V){\to}\mathcal{H}(E)$ of a function $f{\in}\mathcal{H}(V)$ is the vector of all weighted discrete derivatives:

$$d_w f = \big((d_w f)(u,v)\big)_{(u,v)\in E} \text{ where for all } (u,v){\in}E \ ,$$

$$(d_w f)(u,v){=}w_{uv}^{1/2}\big(f(v){-}f(u)\big) \ ,$$

and $\partial_v f(u) = (d_w f)(u,v)$ is the *discrete partial derivative* of f. This definition has the following properties for a function defined in a Euclidean space: $\partial_v f(u) = -\partial_u f(v)$, $\partial_u f(u) = 0$ and if $f(u) = f(v)$ then $\partial_v f(u) = 0$. Based on the difference operator, we define two new *weighted morphological directional difference operators*. The weighted morphological *external* and *internal* difference operator are respectively:

$$
\begin{aligned}
&(d_w^+ f)(u,v) = w_{uv}^{1/2}\big(\max\big(f(u), f(v)\big) - f(u)\big) \text{ and}\\
&(d_w^- f)(u,v) = w_{uv}^{1/2}\big(f(u) - \min\big(f(u), f(v)\big)\big) \ ,
\end{aligned} \tag{3}
$$

with the following properties: $(d_w^- f)(u,v) = (d_w^+ f)(v,u)$

$$
\begin{aligned}
&(d_w^+ f)(u,v) = \max\big(0, (d_w f)(u,v)\big),\\
&(d_w^- f)(u,v) = -\min\big(0, (d_w f)(u,v)\big) \ .
\end{aligned}
$$

since one has $\max(a,b) - a = \max(0, a-b)$, $a - \min(a,b) = -\min(0, b-a)$, and $\max(0, a-b) = -\min(0, b-a)$. The two proposed weighted morphological directional difference operators (3) recover classical directional difference operators on unweighted graphs ($w=1$) and extend them to weighted graphs that enables more adaptation in the difference computation. Finally, the corresponding external and internal partial derivatives are: $\partial_v^+ f(u) = (d_w^+ f)(u,v)$ and $\partial_v^- f(u) = (d_w^- f)(u,v)$.

2.2 Weighted morphological gradients

The *weighted gradient* of a function $f \in \mathcal{H}(V)$ at vertex u is the vector of all edge directional derivatives $(\nabla_w f)(u) = \big(\partial_v f(u)\big)_{(u,v)\in E}$. Following this definition, we introduce two *new weighted morphological (internal and external) gradients* based on the internal and external partial derivatives such as

$$
\begin{aligned}
&(\nabla_w^+ f)(u) = \big(\partial_v^+ f(u)\big)_{(u,v)\in E} \text{ and}\\
&(\nabla_w^- f)(u) = \big(\partial_v^- f(u)\big)_{(u,v)\in E} \ .
\end{aligned} \tag{4}
$$

The external gradient of a function is a directional difference operator defined as the difference between an extensive operator and the function (a typical one being the $\max$). Similarly, the internal gradient uses an anti-extensive (the $\min$) operator [6]. In the sequel, we use the $\mathcal{L}_p$-norm of gradients (4) for a given vertex $u \in V$, we have

$$
\begin{aligned}
&\|(\nabla_w^+ f)(u)\|_p = \Big[\sum_{v\sim u} w_{uv}^{p/2} \big|\max\big(0, f(v) - f(u)\big)\big|^p\Big]^{1/p} \text{ and}\\
&\|(\nabla_w^- f)(u)\|_p = \Big[\sum_{v\sim u} w_{uv}^{p/2} \big|\min\big(0, f(v) - f(u)\big)\big|^p\Big]^{1/p} \ ;
\end{aligned} \tag{5}
$$

and for the $\mathcal{L}_\infty$-norm, we have

$$
\begin{aligned}
&\|(\nabla_w^+ f)(u)\|_\infty = \max_{v\sim u}\Big(w_{uv}^{1/2} \big|\max\big(0, f(v) - f(u)\big)\big|\Big) \text{ and}\\
&\|(\nabla_w^- f)(u)\|_\infty = \max_{v\sim u}\Big(w_{uv}^{1/2} \big|\min\big(0, f(v) - f(u)\big)\big|\Big) \ .
\end{aligned} \tag{6}
$$

Same expressions can be obtained for the general weighted gradient. One can note that general definitions presented in this Section are defined on graphs of arbitrary topology. Hence, they can be used to process any discrete regular or irregular data sets that can be represented by a weighted graph. Moreover, local and nonlocal settings are directly handled in these definitions and both are expressed by the graph topology in terms of neighborhood connectivity [4].

2.3 Relations with algebraic morphology

The previously defined external and internal gradients operate on any graph structure. In the sequel, we show that in the particular case of an unweighted ($w=1$) graph and with $p=\infty$, our gradient formulations recover algebraic morphological operators where the structuring element is provided by the graph neighborhood i.e. $B = N(u)$ for all $u \in V$. The $\mathcal{L}_\infty$-norms (6) of the proposed external and internal gradients ∇_w^+ and ∇_w^- recover the classical definition of algebraic morphological external and internal gradients. Indeed, for the external gradient

$$
\begin{aligned}
\|(\nabla_w^+ f)(u)\|_\infty \quad &= \max_{v\sim u}\big(\max\big(0, f(v) - f(u)\big)\big)\\
= \max_{v\sim u}\big(f(u), f(v)\big) - f(u) \quad &= \delta(f)(u) - f(u) \ ,
\end{aligned}
$$

and similarly for the internal one. With these latter relations, we immediately recover the algebraic classical morphological gradient and Laplace operators:

$$
\begin{aligned}
&\|(\nabla_w^+ f)(u)\|_\infty + \|(\nabla_w^- f)(u)\|_\infty = \delta(f)(u) - \varepsilon(f)(u) \text{ and}\\
&\|(\nabla_w^+ f)(u)\|_\infty - \|(\nabla_w^- f)(u)\|_\infty = \delta(f)(u) + \varepsilon(f)(u) - 2f(u) \ .
\end{aligned}
$$

Finally, our formulation recovers classical morphological gradient and Laplace operators and extend them to weighted graphs that define new families of local and nonlocal weighted morphological gradients and Laplace operators.

2.4 Relations with graph boundary

Intuitively from definitions (2), dilation over $\mathcal{A}$ can be interpreted as a growth process that adds vertices from $\partial^+\mathcal{A}$ to $\mathcal{A}$. By duality, erosion over $\mathcal{A}$ can be interpreted as a contraction process that removes vertices from $\partial^-\mathcal{A}$. The decomposition of f into its level sets is denoted $f^l = \chi(f - l)$ where χ is the Heaviside function (a step function). Then, one can prove [8] that for any level set f^l, at vertex $u \in V$, the $\mathcal{L}_p$-norm of the gradient $(\nabla_w f^l)(u)$ can be decomposed as

$$
\|(\nabla_w f^l)(u)\|_p^p = \|(\nabla_w^+ f^l)(u)\|_p^p + \|(\nabla_w^- f^l)(u)\|_p^p \ .
$$

Moreover, we can also deduce that

$$
\|(\nabla_w f^l)(u)\|_p = \begin{cases} \|(\nabla_w^+ f^l)(u)\|_p \text{ if } u \in \partial^+\mathcal{A}^l,\\ \|(\nabla_w^- f^l)(u)\|_p \text{ if } u \in \partial^-\mathcal{A}^l. \end{cases}
$$

3 Non local graph morphology

Starting from PDEs-based dilation and erosion formulations (1), we define the discrete analogue of such definitions and obtain the following expressions over graphs. For a given initial function $f^0 \in \mathcal{H}(V)$: $\frac{\partial\delta(f)(u)}{\partial t} = \partial_t f(u) = +\|(\nabla_w^+ f)(u)\|_p$ and $\frac{\partial\varepsilon(f)(u)}{\partial t} = \partial_t f(u) = -\|(\nabla_w^- f)(u)\|_p \quad \forall u \in V$, with the initial condition $\partial_{t=0} f = f^0$ (f is a modified version of f^0) and ∇_w^+ and ∇_w^- are the weighted discrete morphological gradients defined in (4).

3.1 Dilation and erosion processes

As for the continuous case, a simple variational definition of dilation applied to f^l can be interpreted as maximizing a surface gain proportional to the gradient norm $+\|\nabla_w f^l\|_p$. Similarly, erosion can be viewed as minimizing a surface gain proportional to $-\|\nabla_w f^l\|_p$. Dilation of f^l on $\mathcal{A}^l$ corresponds to only consider the external boundary set $\partial^+\mathcal{A}^l$ and can be expressed by $\partial_t f^l(u){=}{+}\|(\nabla_w^+ f^l)(u)\|_p$ where the gradient $\|(\nabla_w f^l)(u)\|_p$ is reduced to the external gradient $\|(\nabla_w^+ f^l)(u)\|_p$. Similarly, erosion of f^l on $\mathcal{A}^l$ can be defined by $\partial_t f^l(u){=}{-}\|(\nabla_w^- f^l)(u)\|_p$. Finally, by extending these two processes for all the levels of f, we can naturally consider the following two families of dilation and erosion. These two processes are parameterized by p and w over any weighted graph $G = (V, E, w)$: $\delta(f)(u) = \partial_t f(u) = +\|(\nabla_w^+ f)(u)\|_p$ and $\varepsilon(f)(u) = \partial_t f(u) = -\|(\nabla_w^- f)(u)\|_p$. To solve these latter dilation and erosion processes, on the contrary to the PDEs case, no spatial discretization is needed thanks to derivatives that are directly expressed in a discrete form. Then, by using discretization in time, and with the usual notation $f(u,n) \approx f(u, n\Delta t)$, the general iterative schemes for dilation and erosion, can be defined at time $n{+}1$, for all $u \in V$, as

$$f^{n+1}(u){=}f^n(u){+}\Delta t\|(\nabla_w^+ f)^n(u)\|_p \text{ and} \quad (7)$$

$$f^{n+1}(u){=}f^n(u){-}\Delta t\|(\nabla_w^- f)^n(u)\|_p \;. \quad (8)$$

The initial condition is $f^{(0)}{=}f^0$ where $f^0{\in}\mathcal{H}(V)$ is the initial function defined on the graph vertices. If dilation is considered and with the corresponding gradient norms, (7) becomes for $0{<}p{<}{+}\infty$

$$f^{n+1}(u) \overset{(5)}{=} f^n(u){+}\Delta t\Big(\sum_{v\sim u} w_{uv}^{p/2}\Big|\max\Big(0,\beta_{uv}^n\Big)\Big|^p\Big)^{\frac{1}{p}} \quad (9)$$

and for $p{=}\infty$

$$f^{n+1}(u) \overset{(6)}{=} f^n(u){+}\Delta t \max_{v\sim u}\Big(w_{uv}^{1/2}\Big|\max\Big(0,\beta_{uv}^n\Big)\Big|\Big) \;, \quad (10)$$

where $\beta_{uv}^n{=}f^n(v){-}f^n(u)$. At each step of the algorithms, the new value at vertex u only depends on its value at step n and the existing values in its neighborhood. The proposed dilation and erosion formulations operate on any graph structures. It can be shown that with an adapted graph topology and an appropriated weight function, the proposed methodology for dilation and erosion is linked to well-known methods defined in the context of image processing. With an unweighted ($w = 1$) 4-adjacency grid graph associated with a grayscale image defined as function $f^0{:}V{\subset}\mathbb{Z}^2{\rightarrow}\mathbb{R}$, our formulation of dilation ($p = 2$) recovers the exact Osher-Sethian [5] upwind first order discretization scheme of dilation. The proposed dilation expression also recovers the classical algebraic flat morphological dilation formulation over graphs. In the case where $p{=}\infty$ with a constant discretization time $\Delta t{=}1$ and a constant weight function ($w = 1$), (10) becomes for $u{\in}V$:

$$f^{n+1}(u){=}f^n(u){+}\max_{v\sim u}\Big(\max\big(0,\beta_{uv}^n\big)\Big) = \max_{v\sim u}\big(f^n(v), f^n(u)\big). \quad (11)$$

In that case, the structuring element is provided by both the graph topology and the vertices' neighborhoods. For instance, if we consider a 8-adjacency image grid graph, it is equivalent to a dilation by a square structuring element of size $3{\times}3$. It is important to note that for the case of weighted graphs, the proposed morphological operators are operators that adapt their behavior according to the importance of weights with their neighbors.

4 Results

Figure 1 presents dilation and closing of an original scalar grayscale image considered as a function $f^0{:}\,V \subset \mathbb{Z}^2{\rightarrow}\mathbb{R}$ that defines a mapping from vertices to grayscale pixel values. This example shows the adaptative aspect of our morphological framework. First row of Fig. 1 shows results for the case where $p{=}\infty$, $\Delta t{=}1$. This corresponds to the algebraic processing. In that case, the graph is a 25-adjacency grid graph (equivalent to a circle structuring element of radius 2, denoted as G_2). The last three next rows show results for the case where $p{=}2$. Second row of Fig. 1 presents results for an unweighted ($w{=}1$) 4-adjacency grid graph (denoted as G_0. This case corresponds to PDEs-based morphological processing. Difference between second and third row is that in the latter case, we use a weighted ($w{=}g_1$) graph instead of an unweighted one. These results show the benefits of weights that enable to better preserve edge information as compared to the unweighted cases. Fourth row presents nonlocal patch-based results. In this example the graph is a 25-adjacency grid graph weighted by function g_1 associated with patches of size $5{\times}5$ as pixel features. These results clearly show that this morphological processing better preserves image components such as edges, fine structures and repetitive elements. Figure 2 shows application of morphological alternated sequential filters on 3D data (a mesh, see Fig. 2(a)). The graph corresponds here to the natural graph representation of meshes where each vertex is associated to the 3D spatial coordinates of each mesh point. This latter graph is weighted by function g_1. Figures 2(b) and 2(c) show the evolution of the simplification for iterations $n{=}10$ and $n{=}50$, respectively. It is important to note that the number of vertices does not change during the filtering process and vertices have moved to similar spatial coordinates. Figure 3 shows examples of real world databases processing within our morphological framework. The first row of this figure shows the original data sets. Images come from the United States Postal Service (USPS) handwritten database. This database consists in grayscale images scanned from digits 0 to 9. Each image is of size $16{\times}16$. In these experiments, we use three randomly subsampled test sets of 100 samples for each database. Graphs associated with the original data sets are k-NN graphs weighted with function g_2. It is important to note that each vertex of the graphs corresponds to an image sample and is described by a 256-dimensions ($\mathbb{R}^{16\times16}$) feature vector. Dilation, erosion and opening operations are respectively shown in Figure 3. One can note the filtering and simplification effects of opening operation on the data. Indeed, this operation tends to reduce the data to new artificial and uniform image samples. Finally, such morphological operations on databases can be used as pre-processing steps for data mining purposes.

Figure 1: Morphological Image Processing with graphs (see text for details).

5 Conclusion

In this paper, a novel formalism for nonlocal morphology on graphs has been proposed. This provides a framework that extends PDEs-based methods to discrete local and nonlocal schemes. Moreover, this enables to process by morphological means any high dimensional unorganized multivariate data represented by a weighted graph. The potentiality and the flexibility of our approach has been illustrated for the morphological processing of organized data (regular domain) and unorganized data (irregular domain).

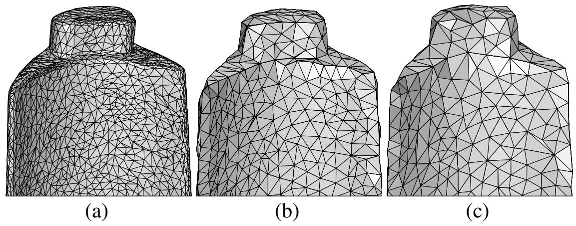

Figure 2: Morphological Processing of a Mesh Graph (see text for details).

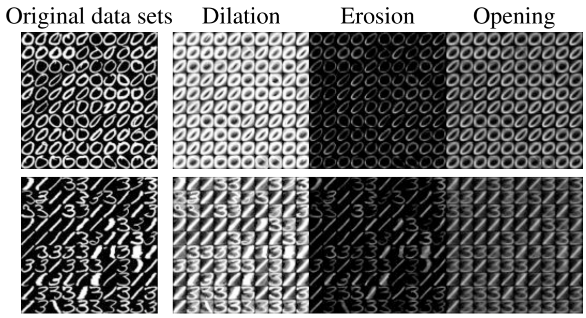

Figure 3: Morphological Procesing of an Image Database Graph (see text for details).

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